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$\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifold models of non factorisable six dimensional toroidal manifolds

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Abstract: We discuss heterotic strings on $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of non factorisable six-tori. Although the number of fixed tori is reduced as compared to the factorisable case, Wilson lines are still needed for the construction of three generation models. An essential new feature is the straightforward appearance of three generation models with one generation per twisted sector. We illustrate our general arguments for the occurrence of that property by an explicit example. Our findings give further support for the conjecture that four dimensional heterotic strings formulated at the free fermionic point are related to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

Keywords: Superstring Vacua, Superstrings and Heterotic Strings.

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Contents

1. Introduction

String theory continues to provide the only viable framework for the unification of all the known fundamental matter and interactions. Pivotal steps in this development were the discoveries of anomaly cancellation in ten dimensions [1] and the subsequent realisation that compactification to four dimensions yields the structures anticipated in Grand Unified Theories [2]. Further studies revealed that the different string theories in ten dimensions, together with eleven dimensional supergravity, probe, in fact, a single underlying theory [3]. The downside to these promising developments is that the compactification of string theory to four dimensions yields a large number of possible vacua and, a priori, there does not exist an apparent mechanism that selects among them.

On the other hand, the data extracted from collider, and other, experiments yielded the Standard Particle Model as the correct accounting of all the observed data in the accessible energy range. Furthermore, the particle physics data is compatible with the hypothesis that the renormalisable Standard Model remains unaltered up to a large energy scale and that the particle spectrum is embedded in a Grand Unified Theory [4]. Most appealing in this context is SO(10) unification, in which each Standard Model generation is embedded in a single 16 spinorial representation of SO(10). However, elucidating further the properties of the Standard Model spectrum, such as the existence of flavor, necessitates the unification of the Standard Particle Model with gravity.

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Spinorial representations of SO(10) are obtained in the heterotic string [5], but not in the other perturbative limits of the underlying non perturbative theory. Thus, maintaining the SO(10) embedding of the Standard Model spectrum necessitates that we study compactifications of the heterotic string on internal six dimensional manifolds. However, preserving the SO(10) embedding of the Standard Model spectrum in three generation string vacua has proven to be an intricate challenge.

A particular class of four dimensional string models, that do yield three generations with the canonical SO(10) embedding, are the heterotic string models in the so-called free fermionic formulation [6]. The free fermionic formalism, however, is constructed directly in four dimensions and the notion of a compactified manifold is a priori lost. The free fermionic models, in general, correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds at special points in the moduli space. In specific cases one can make the correspondence precise [7], while in the general case it is anticipated.

The correspondence of the free fermionic models with $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds therefore hinges on these compactifications as the interesting ones to explore for phenomenological purposes. Such studies were indeed pursued over the last few years [8]. However, only with partial success, in the sense that, while three generation models were found they, did not yield the standard SO(10) embedding of the Standard Model spectrum. Specifically, they did not produce models in which the weak hypercharge has the canonical SO(10) normalisation. The results of [8] suggest, however, that there is a vast number of three generation models if discrete Wilson lines [9] are included into the construction. Indeed, on factorisable lattices, the number of possible independent Wilson lines is six. The number of models is large such that even a computer aided classification would be a quite formidable task. More intuition from elsewhere is needed.

It is therefore drawn upon us to seek further insight from the free fermionic models. The primary property of the free fermionic formulation is of being formulated a priori at a maximally symmetric point in the toroidal lattice moduli space. The distinct feature is that the enhanced lattice at the free fermionic point in the moduli space is a genuine T^6 lattice and is not factorisable to a product of three T^2 tori. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifolds that have been constructed to date are all on factorisable lattices.

In this paper, we therefore extend these studies to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on non factorisable lattices. We elucidate further the connection between the free fermionic models and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds by demonstrating how the $\mathbb{Z}_2 \times \mathbb{Z}_2$ fixed point structure is modified on non factorisable orbifolds and, in the case of SO(12) lattice, matches that of the free fermionic model. While this result has been known for some time, our discussion here is novel in the sense that we demonstrate how the total number of fixed points on the SO(12) lattice is reduced due to the SO(12) root lattice identifications. We then analyse $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifolds on other non factorisable lattices. We include Wilson lines in the analysis and present one three generations model in this class. Our paper therefore represents important advance in elucidating the geometrical correspondence of the free fermionic models, as well as opening the investigation of new classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

The paper is organised as follows. In section two we review the free fermionic construction, focusing especially on the features enabling a systematic approach to the search

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for realistic models. The rest of the paper is then devoted to orbifold compactifications of the heterotic string. These construction have been developed since the 1980s, e.g. in [9 – 15]. More recent studies focus on the geometric explanation of phenomenologically relevant properties [8, 16 – 23].

Section three addresses the geometry of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of non factorisable tori. In particular the number of fixed tori is explicitly counted and computed via a Lefschetz fixed point theorem for various examples. Further, the Euler number for all examples is determined. In section four the geometric data are related to the number of generations for standard embeddings. As predicted by index theorems, the net number of generations always equals half the Euler number, whereas the number of fixed tori provides additional information about the number of generations and anti-generations. In section five, we use Wilson lines in order to construct an explicit three generation model with gauge group SO(10). In contrast to the case of a T^6 factorising into three T^2 factors, one can easily find models with one generation per twisted sector for non factorisable T^6. The reason is, that for the latter, cycles exist which are invariant under none of the non trivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements. More details are given in section five. Finally, we finish with some concluding remarks in section six.

2. Realistic free fermionic models

In this section we briefly discuss the general structure of the free fermionic models and their correspondence with $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifolds. The notation and details of the construction of these models are given elsewhere [24 – 29].

In the free fermionic formulation, the 4-dimensional heterotic string, in the light-cone gauge, is described by 20 left moving and 44 right moving real fermions. A large number of models can be constructed by choosing different phases picked up by fermions $(f_A, A = 1, \ldots, 64)$ when transported along the torus non-contractible loops. Each model corresponds to a particular choice of fermion phases consistent with modular invariance that can be generated by a set of basis vectors $b_i, i = 1, \ldots, n$

$$b_i = \{b_i(f_1), b_i(f_2), b_i(f_3) \ldots\}$$

describing the transformation properties of each fermion

$$f_A \to -e^{i\pi b_i(f_A)}\ f_A, \qquad A = 1, \ldots, 64 \tag{2.1}$$

The basis vectors span a space Ξ which consists of 2^n sectors that give rise to the string spectrum. Each sector is given by

$$\alpha = \sum m_i b_i, \qquad m_i = 0, \ldots, N_i \tag{2.2}$$

The spectrum is truncated by a generalised GSO projection whose action on a string state $|S\rangle$ is

$$e^{i\pi b_i \cdot F_S}|S\rangle = \delta_S\ c\begin{bmatrix} S \\ b_i \end{bmatrix}^* |S\rangle, \tag{2.3}$$

where F_S is the fermion number operator and $\delta_S = \pm 1$ is the spacetime spin statistics index. Different sets of projection coefficients $c\binom{S}{b_i}$ consistent with modular invariance give rise to different models. A model is defined uniquely by a set of basis vectors b_i, $i = 1, \ldots, n$ and a set of $2^{n(n-1)/2}$ independent projection coefficients $c\binom{b_i}{b_j}$, $i > j$.

The boundary condition basis set defining a typical realistic free fermionic heterotic string model is constructed in two stages. The first stage consists of the NAHE set, which is composed of five boundary condition basis vectors, $\{1, S, b_1, b_2, b_3\}$ [30, 28]. The gauge group induced by the NAHE set is $\mathrm{SO}(10) \times \mathrm{SO}(6)^3 \times \mathrm{E}_8$ with $N = 1$ supersymmetry. The space-time vector bosons that generate the gauge group arise from the Neveu-Schwarz sector and from the sector $\xi_2 \equiv 1 + b_1 + b_2 + b_3$. The Neveu-Schwarz sector produces the generators of $\mathrm{SO}(10) \times \mathrm{SO}(6)^3 \times \mathrm{SO}(16)$. The ξ_2-sector produces the spinorial 128 of SO(16) and completes the hidden gauge group to E_8. The NAHE set divides the internal world-sheet fermions in the following way: $\bar{\phi}^{1,\ldots,8}$ generate the hidden E_8 gauge group, $\bar{\psi}^{1,\ldots,5}$ generate the SO(10) gauge group, and $\{\bar{y}^{3,\ldots,6}, \bar{\eta}^1\}$, $\{\bar{y}^1, \bar{y}^2, \bar{\omega}^5, \bar{\omega}^6, \bar{\eta}^2\}$, $\{\bar{\omega}^{1,\ldots,4}, \bar{\eta}^3\}$ generate the three horizontal SO(6) symmetries. The left-moving $\{y, \omega\}$ states are divided into $\{y^{3,\ldots,6}\}$, $\{y^1, y^2, \omega^5, \omega^6\}$, $\{\omega^{1,\ldots,4}\}$ and χ^{12}, χ^{34}, χ^{56} generate the left-moving $N = 2$ world-sheet supersymmetry. At the level of the NAHE set, the sectors b_1, b_2 and b_3 produce 48 multiplets, 16 from each, in the 16 representation of SO(10). The states from the sectors b_j are singlets of the hidden E_8 gauge group and transform under the horizontal $\mathrm{SO}(6)_j$ ($j = 1, 2, 3$) symmetries. This structure is common to a large class of quasi-realistic free fermionic models.

The second stage of the construction consists of adding to the NAHE set three (or four) additional basis vectors. These additional vectors reduce the number of generations to three, one from each of the sectors b_1, b_2 and b_3, and simultaneously break the four dimensional gauge group. The assignment of boundary conditions to $\{\bar{\psi}^{1,\ldots,5}\}$ breaks SO(10) to one of its subgroups $\mathrm{SU}(5) \times \mathrm{U}(1)$ [24], $\mathrm{SO}(6) \times \mathrm{SO}(4)$ [26], $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^2$ [25, 27], $\mathrm{SU}(3) \times \mathrm{SU}(2)^2 \times \mathrm{U}(1)$ [29] or $\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ [31]. Similarly, the hidden E_8 symmetry is broken to one of its subgroups, and the flavor $\mathrm{SO}(6)^3$ symmetries are broken to $\mathrm{U}(1)^n$, with $3 \leq n \leq 9$. For details and phenomenological studies of these three generation string models we refer interested readers to the original literature and review articles [32].

The correspondence of the free fermionic models with the orbifold construction is illustrated by extending the NAHE set, $\{1, S, b_1, b_2, b_3\}$, by at least one additional boundary condition basis vector [7]

$$\xi_1 = (0, \ldots, 0 | \underbrace{1, \ldots, 1}_{\bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}}, 0, \ldots, 0) \ . \tag{2.4}$$

With a suitable choice of the GSO projection coefficients the model possesses an $\mathrm{SO}(4)^3 \times \mathrm{E}_6 \times \mathrm{U}(1)^2 \times \mathrm{E}_8$ gauge group and $N = 1$ space-time supersymmetry. The matter fields include 24 generations in the 27 representation of E_6, eight from each of the sectors $b_1 \oplus b_1 + \xi_1$, $b_2 \oplus b_2 + \xi_1$ and $b_3 \oplus b_3 + \xi_1$. Three additional 27 and $\overline{27}$ pairs are obtained from the Neveu-Schwarz $\oplus$ ξ_1 sector.

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To construct the model in the orbifold formulation one starts with the compactification on a torus with nontrivial background fields [33]. The subset of basis vectors

$$\{1, S, \xi_1, \xi_2\} \tag{2.5}$$

generates a toroidally-compactified model with $N = 4$ space-time supersymmetry and $\mathrm{SO}(12) \times \mathrm{E}_8 \times \mathrm{E}_8$ gauge group. The same model is obtained in the geometric (bosonic) language by tuning the background fields to the values corresponding to the SO(12) lattice. The metric of the six-dimensional compactified manifold is then the Cartan matrix of SO(12), while the antisymmetric tensor is given by

$$b_{ij} = \begin{cases} g_{ij} & ;\ i > j, \\ 0 & ;\ i = j, \\ -g_{ij} & ;\ i < j. \end{cases} \tag{2.6}$$

When all the radii of the six-dimensional compactified manifold are fixed at $R_I = \sqrt{2}$, it is seen that the left- and right-moving momenta

$$P^I_{R,L} = \left[m_i - \frac{1}{2}(B_{ij} \pm G_{ij}) n_j \right] e^{I*}_i \tag{2.7}$$

reproduce the massless root vectors in the lattice of SO(12). Here $e^i = \{e^I_i\}$ are six linearly-independent vielbeins normalised so that $(e_i)^2 = 2$. The e^{I*}_i are dual to the e_i, with $e^*_i \cdot e_j = \delta_{ij}$.

Adding the two basis vectors b_1 and b_2 to the set (2.5) corresponds to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model with standard embedding. Starting from the $N = 4$ model with $\mathrm{SO}(12) \times \mathrm{E}_8 \times \mathrm{E}_8$ symmetry, and applying the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist on the internal coordinates, reproduces the spectrum of the free-fermion model with the six-dimensional basis set $\{1, S, \xi_1, \xi_2, b_1, b_2\}$ [7]. The Euler characteristic of this model is 48 with $h_{11} = 27$ and $h_{21} = 3$.

It is noted that the effect of the additional basis vector ξ_1 of eq. (2.4), is to separate the gauge degrees of freedom, spanned by the world-sheet fermions $\{\bar\psi^{1,\ldots,5}, \bar\eta^1, \bar\eta^2, \bar\eta^3, \bar\phi^{1,\ldots,8}\}$, from the internal compactified degrees of freedom $\{y, \omega | \bar y, \bar\omega\}^{1,\ldots,6}$. In the realistic free fermionic models this is achieved by the vector 2γ [7], with

$$2\gamma = (0, \ldots, 0 | \underbrace{1, \ldots, 1}_{\bar\psi^{1,\ldots,5}, \bar\eta^{1,2,3} \bar\phi^{1,\ldots,4}}, 0, \ldots, 0), \tag{2.8}$$

which breaks the $\mathrm{E}_8 \times \mathrm{E}_8$ symmetry to $\mathrm{SO}(16) \times \mathrm{SO}(16)$. The $Z_2 \times Z_2$ twist induced by b_1 and b_2 breaks the gauge symmetry to $\mathrm{SO}(4)^3 \times \mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SO}(16)$. The orbifold still yields a model with 24 generations, eight from each twisted sector, but now the generations are in the chiral 16 representation of SO(10), rather than in the 27 of E_6. The same model can be realised [34] with the set $\{1, S, \xi_1, \xi_2, b_1, b_2\}$, by projecting out the $16 \oplus \overline{16}$ from the ξ_1-sector taking

$$c \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \to -c \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{2.9}$$

This choice also projects out the massless vector bosons in the 128 of SO(16) in the hidden-sector E_8 gauge group, thereby breaking the $E_6 \times E_8$ symmetry to $SO(10) \times U(1) \times SO(16)$. We can define two $N = 4$ models generated by the set (2.5), Z_+ and Z_-, depending on the sign in eq. (2.9). The first, say Z_+, produces the $E_8 \times E_8$ model, whereas the second, say Z_-, produces the $SO(16) \times SO(16)$ model. However, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist acts identically in the two models, and their physical characteristics differ only due to the discrete torsion eq. (2.9).

Several remarks are important to note at this stage. The first is that a priori the point at which the internal dimensions are realised as free fermions on the world-sheet is a maximally symmetric point with an enhanced SO(12) lattice. This lattice is a priori not factorisable to a product of three T^2 lattices, but is rather a non factorisable T^6 lattice. The phenomenologically appealing properties of the free fermionic models and their relation to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds provide the clue that we might gain further insight into the properties of this class of quasi-realistic string compactifications by constructing $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on enhanced non factorisable lattices.

It is further important to note that the free fermionic formalism enables a classification of a wide range of $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifolds. The correspondence illustrated above exhibits in detail the correspondence of a single free fermionic model with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold in a specific case. The correspondence in the general case is seen by analysing the respective partition functions and it is anticipated that for every $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold one can write a corresponding free fermionic partition function at special points in the moduli space, and hence obtain a representation in terms of free fermion boundary condition basis vectors. In turn, using the free fermion formalism the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds is carried out by specifying a set of boundary condition basis vectors and one-loop GSO coefficients. The details of this classification for type II strings were carried in [35] and for the heterotic string in [36].

The free fermionic formalism provides useful means to classify and analyse $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifolds at special points in the moduli space. The drawbacks of this approach is that the analysis is carried out at special points in the moduli space and the geometric view of the underlying compactifications is hindered. On the other hand, the geometric picture may be instrumental for examining other questions of interest, such as the dynamical stabilisation of the moduli fields and the moduli dependence of the Yukawa couplings. In the following we turn to analyse $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on non factorisable toroidal manifolds.

3. $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of non factorisable tori

In this section we discuss the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of non factorisable T^6 tori. We use a geometrically intuitive picture that facilitates understanding the fixed point structure on these lattices. We pay special attention to the identifications by the non factorisable root lattice vectors and how the number of fixed tori is affected. Our discussion illuminates a long standing puzzle in the orbifold community in regard to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold correspondence of the free fermionic models.

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In the following we will exemplarily discuss a set of compactifications on non factorisable six-tori. These will illustrate the main features of such compactifications. For more systematic studies (from slightly different perspectives) see [7, 11, 38]. We will determine the number of fixed tori once by explicit counting and once via a Lefschetz fixed point theorem. In addition, we give the Euler number of the orbifold. For the standard embedding of the orbifold action into the gauge group the net number of generations will be fixed by the Euler number, whereas the number of generations and anti-generations depends also on the number of fixed tori. (For different orbifolds this is discussed in [39].)

We specify the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold action on a set of six Cartesian coordinates $x^1, \ldots, x^6$ of the compact space as follows:

$$\begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix} \to \theta_1 \begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix}, \quad \text{with } \theta_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

and

$$\begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix} \to \theta_2 \begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix}, \quad \text{with } \theta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.2}$$

where θ_1 and θ_2 are the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3.1 SO(12) lattice

As a first example, we consider the case that T^6 is obtained by compactifying $\mathbb{R}^6$ on an SO(12) root lattice whose basis vectors are given by the simple roots

$$\begin{aligned} e_1 &= (1, -1, 0, 0, 0, 0)\,, \\ e_2 &= (0, 1, -1, 0, 0, 0)\,, \\ e_3 &= (0, 0, 1, -1, 0, 0)\,, \\ e_4 &= (0, 0, 0, 1, -1, 0)\,, \\ e_5 &= (0, 0, 0, 0, 1, -1)\,, \\ e_6 &= (0, 0, 0, 0, 1, 1)\,. \end{aligned} \tag{3.3}$$

The orbifold action, by e.g. (3.2), leaves sets of points invariant, i.e. these points differ from their orbifold image by an SO(12) root lattice shift. For our particular choice of the orbifold action these sets appear as two dimensional fixed tori. In the following we will list 16 such two-tori and afterwards argue that some of these 16 tori are identical. That will leave us in the end with eight different two-tori.

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The trivial fixed torus is given as the set

$$\left\{(x,y,0,0,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}. \tag{3.4}$$

The compactification lattice Λ^2 is generated by the vectors $(1,1)$ and $(1,-1)$. This can be verified by writing

$$(x,y,0,0,0,0)=xe_1+(x+y)\left(e_2+e_3+e_4+\frac{1}{2}e_5+\frac{1}{2}e_6\right)$$

and identifying minimal shifts in (x,y) shifting the coefficients in front of lattice vectors by integers.

Now, consider the fixed torus

$$\left\{(x,y,1,0,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}. \tag{3.5}$$

Points on that torus differ from their image point by the lattice vector $(0,0,2,0,0,0)$. The position of the 1 entry can be altered within the last four components by adding SO(12) root vectors, e.g. $(0,0,-1,1,0,0)$.

Next there are fixed tori of the form

$$\left\{\left(x,y,\underline{\frac{1}{2},\frac{1}{2}},0,0\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}, \tag{3.6}$$

where the underlined entries can be permuted. Points on these fixed tori differ from their orbifold image by an SO(12) root, e.g. $(0,0,1,1,0,0)$. There are

$$\binom{4}{2}=6$$

such fixed two-tori.

Very similar fixed tori are

$$\left\{\left(x,y,\underline{\frac{1}{2},-\frac{1}{2}},0,0\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}, \tag{3.7}$$

where the position of the minus sign can be changed by lattice shifts ($(1/2,-1/2)+(-1,1)=(-1/2,1/2)$). This yields another set of six fixed tori.

Finally, there are the fixed tori

$$\left\{\left(x,y,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\} \tag{3.8}$$

and

$$\left\{\left(x,y,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}. \tag{3.9}$$

So, altogether we have listed 16 fixed tori. Now, we are going to argue that some of them are equivalent. Consider the fixed torus (3.5) and add the SO(12) root vector $(1,0,-1,0,0,0)$. This yields an equivalent expression for (3.5)

$$\left\{(x+1,y,0,0,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}. \tag{3.10}$$

But this is the same fixed torus as (3.4), merely the origin for the x coordinate has been shifted by one. Similar arguments show that the tori in (3.6) and (3.7) as well as the tori (3.8) and (3.9) are mutually equivalent. So, finally we are left with eight inequivalent fixed tori.

For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold we add another $\mathbb{Z}_2$ action θ_1 (3.1). For this $\mathbb{Z}_2 \times \mathbb{Z}_2$ action we obtain eight fixed tori under the action of θ_1, eight fixed tori under the action of θ_2 and eight fixed tori under the action of $\theta_1\theta_2$. Hence, the total number of fixed tori is 24.

After the somewhat tricky explicit counting we would like to confirm our result by using mathematical fixed point theorems. For the case of fixed tori the adequate Lefschetz fixed point theorem for the number of fixed tori (# FT) reads [40][1]

$$\#FT = \left| \frac{N}{(1-\theta)\,\Lambda} \right|. \tag{3.11}$$

On the rhs the index of a lattice quotient appears. The lattice in the denominator is a sub-lattice of the lattice in the numerator and the index counts how often the fundamental cell of the finer lattice fits into the one of the coarser one. In our case the coarser lattice is $(1-\theta)\,\Lambda$, which is a sub-lattice of the compactification lattice Λ, obtained by projecting with $(1-\theta)$. The finer lattice is N, which is the lattice normal to the sub-lattice left invariant by the action of θ. A more handy version of (3.11) is

$$\#FT = \frac{\mathrm{vol}\,((1-\theta)\,\Lambda)}{\mathrm{vol}\,(N)}, \tag{3.12}$$

where the volumes of the fundamental cells appear on the rhs. These can be easily determined by computing the induced metrics on the corresponding lattices. Now let us illustrate the theorem at the case of θ_2 fixed tori in the SO(12) lattice. First, we note that

$$1-\theta_2 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&2&0&0&0 \\ 0&0&0&2&0&0 \\ 0&0&0&0&2&0 \\ 0&0&0&0&0&2 \end{pmatrix}. \tag{3.13}$$

A basis in the $(1-\theta_2)\,\Lambda$ lattice is

$$(1-\theta_2)\,\Lambda: \quad \begin{array}{l} (0,0,2,0,0,0)\,, \\ (0,0,0,2,0,0)\,, \\ (0,0,0,0,2,0)\,, \\ (0,0,0,0,0,2)\,, \end{array} \tag{3.14}$$

where e.g. the first basis vector originates from the SO(12) root $(1,0,1,0,0,0)$. Hence, the induced metric is four times a four by four unit matrix and the volume of the fundamental cell can be obtained as the square root of its determinant

$$\mathrm{vol}\,((1-\theta_2)\,\Lambda) = 16. \tag{3.15}$$

[1] In our coordinate system the orbifold action is represented by symmetric matrices and in that case (3.11) is equivalent to the expression given in [41] where a different calculation is described.

The θ_2 invariant lattice is spanned by $(1,1,0,0,0,0)$ and $(1,-1,0,0,0,0)$. Hence, the normal lattice is generated by simple SO(8) roots

$$N: \begin{array}{c} (0,0,1,-1,0,0)\,, \\ (0,0,0,1,-1,0)\,, \\ (0,0,0,1,-1)\,, \\ (0,0,0,0,1,1)\,. \end{array} \tag{3.16}$$

The induced metric is the SO(8) Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}. \tag{3.17}$$

The square root of the determinant of the induced metric provides the volume

$$\mathrm{vol}\,(N) = 2. \tag{3.18}$$

Plugging (3.15) and (3.18) into (3.12), we find that the number of θ_2 fixed tori is eight, confirming our earlier result, obtained by explicit counting. The calculation for the other $\mathbb{Z}_2 \times \mathbb{Z}_2$ non trivial elements θ_1 and $\theta_1\theta_2$ is a straightforward modification of the presented calculation. Each element leaves eight tori fixed, yielding a total of 24 fixed tori.

Finally, we would like to compute the Euler number of the given orbifold. The general expression for the Euler number χ is [10]

$$\chi = \frac{1}{|G|} \sum_{[g,h]=0} \chi_{g,h}, \tag{3.19}$$

where $|G|$ is the order of the orbifold group with elements g, h and $\chi_{g,h}$ is the number of points which are simultaneously fixed under the action of g and h. (Here, it is important that these are really points, if e.g. $g = 1$ then $\chi_{1,h}$ provides the number of tori fixed under h, which do not contribute to the Euler number.)

Now, let us identify points which are fixed under θ_1 and θ_2 in the case that the compactification is on the SO(12) root lattice. One finds 32 such points:

$$\begin{array}{ll} (0,0,0,0,0,0) & 1 \text{ point}, \\ (1,0,0,0,0,0) & 1 \text{ point}, \\ \left(\frac{1}{2},\pm\frac{1}{2},0,0,0,0\right) & +S_3 \ 6 \text{ points}, \\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\pm\frac{1}{2},0,0\right) & +S_3 \ 6 \text{ points}, \\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}\right) & 2 \text{ points}, \\ \left(\underline{\frac{1}{2},0},\underline{\frac{1}{2},0},\underline{\pm\frac{1}{2},0}\right) & 16 \text{ points}, \end{array} \tag{3.20}$$

where "$+S_3$" stands for adding vectors obtained by permuting the first, second and third pair of entries and underlined entries can also be permuted. So, altogether, we find

$$\chi_{\theta_1,\theta_2} = 32. \tag{3.21}$$

This is four times the number of tori fixed under θ_2. The counting for the other five combinations of non-trivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements is the same and we end up with

$$\chi = 48, \tag{3.22}$$

which is twice the number of fixed tori.

3.2 $\mathrm{SO}(6)^2$ lattice — first example

Now, we choose for the T^6 compactification an $\mathrm{SO}(6)^2$ lattice with basis vectors

$$\begin{aligned} e_1 &= (1,-1,0,0,0,0)\,, \\ e_2 &= (0,1,-1,0,0,0)\,, \\ e_3 &= (0,1,1,0,0,0)\,, \\ e_4 &= (0,0,0,1,-1,0)\,, \\ e_5 &= (0,0,0,0,1,-1)\,, \\ e_6 &= (0,0,0,0,1,1)\,. \end{aligned} \tag{3.23}$$

The determination of the fixed tori under θ_2 is very similar to the previous case. First, we point out the differences and then list the fixed tori. The fixed torus

$$\left\{(x,y,0,0,0,1)\,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{3.24}$$

is not equivalent to

$$\left\{(x,y,0,0,0,0)\,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{3.25}$$

since there is no lattice vector like $(1,0,0,0,0,-1)$ in the $\mathrm{SO}(6)^2$ lattice. On the other hand, the fixed torus

$$\left\{(x,y,1,0,0,0)\,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\} \tag{3.26}$$

is equivalent to (3.25) (via a shift with $(1,0,-1,0,0,0)$ and a reparameterisation of the fixed torus). It is not equivalent to (3.24) because the SO(12) lattice vector $(0,0,-1,0,0,1)$ is not in the $\mathrm{SO}(6)^2$ lattice. Taking this kind of arguments into account one finds the following eight fixed tori under the action of θ_2 (3.2)

$$\left\{(x,y,0,0,0,0)\,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \quad 1 \text{ fixed torus}, \tag{3.27}$$

$$\left\{(x,y,0,0,0,1)\,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \quad 1 \text{ fixed torus}, \tag{3.28}$$

$$\left\{\left(x,y,0,\underline{\tfrac{1}{2},\tfrac{1}{2}},0\right)\,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \quad 3 \text{ fixed tori}, \tag{3.29}$$

$$\left\{\left(x,y,0,\underline{\tfrac{1}{2},-\tfrac{1}{2}},0\right)\,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \quad 3 \text{ fixed tori}, \tag{3.30}$$

where underlined entries can be again permuted and the position of the minus sign within the last three entries in (3.30) can be altered by lattice shifts.

By swapping the two SO(6) factors, we obtain the following eight fixed tori under the action of θ_1 (3.1)

$$\left\{(0,0,0,0,x,y)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 1 \text{ fixed torus}, \tag{3.31}$$

$$\left\{(1,0,0,0,x,y)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 1 \text{ fixed torus}, \tag{3.32}$$

$$\left\{\left(\underline{\frac{1}{2},\frac{1}{2}},0,0,x,y\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 3 \text{ fixed tori}, \tag{3.33}$$

$$\left\{\left(\underline{\frac{1}{2},-\frac{1}{2}},0,0,x,y\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 3 \text{ fixed tori}. \tag{3.34}$$

For the action of $\theta_1\theta_2$ the situation is different. As a first difference, we note that the trivially fixed torus is

$$\left\{(0,0,x,y,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\tilde{\Lambda}^2\right\}, \tag{3.35}$$

with a modified lattice $\tilde{\Lambda}^2$ which is generated by $(2,0)$ and $(0,2)$. (The point $(0,0,x,y,0,0)$ $=\frac{x}{2}(e_2+e_3)+\frac{y}{2}(2e_4+e_5+e_6)$ on T^6 is invariant if x or y are shifted by integer multiples of two.) This fixed torus is twice as big as the ones which occurred previously. Now the fixed tori

$$\left\{(1,0,x,y,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\tilde{\Lambda}^2\right\}$$

as well as

$$\left\{(0,0,x,y,1,0)\,\middle|\,x,y\in\mathbb{R}^2/\tilde{\Lambda}^2\right\}$$

are equivalent to the trivial one (by shifts with $(1,0,-1,0,0,0)$ or $(0,0,0,1,-1,0)$ and redefinitions of the torus coordinates x or y). One finds four inequivalent fixed tori:

$$\left\{(0,0,x,y,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\tilde{\Lambda}^2\right\},\quad 1 \text{ fixed torus}, \tag{3.36}$$

$$\left\{\left(\frac{1}{2},\frac{1}{2},x,y,0,0\right)\,\middle|\,x,y\in\mathbb{R}^2/\tilde{\Lambda}^2\right\},\quad 1 \text{ fixed torus}, \tag{3.37}$$

$$\left\{\left(0,0,x,y,\frac{1}{2},\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\tilde{\Lambda}^2\right\},\quad 1 \text{ fixed torus}, \tag{3.38}$$

$$\left\{\left(\frac{1}{2},\frac{1}{2},x,y,\frac{1}{2},\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\tilde{\Lambda}^2\right\},\quad 1 \text{ fixed torus}. \tag{3.39}$$

Hence, we obtain 20 fixed tori in total (eight under the action of θ_1 and θ_2 each and another four from the action of $\theta_1\theta_2$).

Before rederiving this result using the Lefschetz theorem, let us comment on what we found so far. The number of fixed tori under the action of θ_1 and θ_2 is the same as for the SO(12) lattice. In fact, as long as we project only on the e.g. θ_2 invariant states, moduli allowing a deformation into the SO(12) lattice are still present in the spectrum. In particular, the internal metric components, g_{IJ}, with $I,J\geq 3$ are even under the action of θ_1. The compactification lattice can be deformed within the x^3-x^4-x^5-x^6 'plane'. This means that we can continuously deform e_6 in (3.23) into $(0,0,1,1,0,0)$ yielding an SO(12) lattice. Likewise, under θ_1, moduli allowing a deformation within the x^1-x^2-x^3-x^4 'plane' survive the projection. Replacing e_2 with $(0,0,1,-1,0,0)$ can be done in a continuous

way, yielding again an SO(12) lattice. On the other hand, moduli which are even under $\theta_1\theta_2$ allow for deformations within the x^1-x^2-x^5-x^6 and x^3-x^4 planes. Neither of the above replacements yielding the SO(12) lattice corresponds to such a deformation. Thus our findings for the numbers of fixed tori in the different sectors are compatible with possible continuous deformations of the orbifold.

Now, we confirm the result obtained by explicit counting via the Lefschetz fixed point theorem (3.11). First, we compute the number of fixed tori under θ_2. A basis for the $(1-\theta_2)\,\Lambda$ lattice is

$$(1-\theta_2)\,\Lambda: \quad \begin{matrix} (0,0,2,0,0,0)\,, \\ (0,0,0,2,-2,0)\,, \\ (0,0,0,0,2,-2)\,, \\ (0,0,0,0,2,2)\,. \end{matrix} \tag{3.40}$$

The induced metric reads

$$4\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}. \tag{3.41}$$

The square root of its determinant is 32. On the other hand the lattice normal to the invariant lattice is

$$N: \quad \begin{matrix} (0,0,2,0,0,0)\,, \\ (0,0,0,1,-1,0)\,, \\ (0,0,0,0,1,-1)\,, \\ (0,0,0,0,1,1)\,, \end{matrix} \tag{3.42}$$

resulting in the induced metric

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \tag{3.43}$$

whose determinant is 4^2. Hence, the number of tori fixed under θ_2 is $32/4$ =8.

For the number of fixed tori under θ_1 a completely analogous computation yields again eight.

Now, consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ generator $\theta_1\theta_2$ for which

$$(1-\theta_1\theta_2) = \begin{pmatrix} 2&0&0&0&0&0 \\ 0&2&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&2&0 \\ 0&0&0&0&0&2 \end{pmatrix}. \tag{3.44}$$

The lattice $(1-\theta_1\theta_2)\,\Lambda$ is spanned by the following basis

$$(1-\theta_1\theta_2)\,\Lambda: \quad \begin{matrix}(2,0,0,0,0,0)\,,\\ (0,2,0,0,0,0)\,,\\ (0,0,0,0,2,0)\,,\\ (0,0,0,0,0,2)\,.\end{matrix} \tag{3.45}$$

Hence, the induced metric is just four times a four by four identity matrix. The square root of its determinant being 16.

For the normal lattice we find

$$N: \quad \begin{matrix}(1,-1,0,0,0,0)\,,\\ (1,1,0,0,0,0)\,,\\ (0,0,0,0,1,-1)\,,\\ (0,0,0,0,1,1)\,.\end{matrix} \tag{3.46}$$

The metric induced on N is twice a four by four identity matrix. The square root of its determinant is four. Hence, the number of fixed tori in that sector is $16/4 = 4$.

In order to compute the Euler number we have to determine the points which are left invariant under the action of two different non trivial elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. For any such combination we find that the third and fourth entry of the fixed point vector have to vanish, whereas the first two and last two entries can be any of the following four choices

$$(0,0)\,,\ \ (1,0)\,,\ \ \left(\frac{1}{2},\pm\frac{1}{2}\right). \tag{3.47}$$

Hence, we get

$$\chi_{\theta_1,\theta_2} = \chi_{\theta_1,\theta_1\theta_2} = \chi_{\theta_2,\theta_1\theta_2} = 16, \tag{3.48}$$

Note, that this time e.g. χ_{θ_1,θ_2} is not four times the number of tori fixed under θ_1. For the Euler number we obtain

$$\chi = 24, \tag{3.49}$$

which differs from twice the number of fixed tori.

3.3 $\mathrm{SO}(6)^2$ lattice — second example

Now, we consider a version of the previous compactification lattice which exhibits invariance under permuting the x^1-x^2, x^3-x^4 and x^5-x^6 plane, i.e. we choose

$$\begin{aligned} e_1 &= (1,0,-1,0,0,0)\,,\\ e_2 &= (0,0,1,0,-1,0)\,,\\ e_3 &= (0,0,1,0,1,0)\,,\\ e_4 &= (0,1,0,-1,0,0)\,,\\ e_5 &= (0,0,0,1,0,-1)\,,\\ e_6 &= (0,0,0,1,0,1) \end{aligned} \tag{3.50}$$

as a basis for the $\mathrm{SO}(6)^2$ lattice. For lattice vectors in the first (second) SO(6) factor only odd (even) components can be non vanishing.

For that compactification one finds the following four inequivalent fixed tori under the action of θ_2 (3.2):

$$\left\{(x,y,0,0,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 1\text{ fixed torus},\tag{3.51}$$

$$\left\{\left(x,y,\frac{1}{2},0,\frac{1}{2},0\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 1\text{ fixed torus},\tag{3.52}$$

$$\left\{\left(x,y,0,\frac{1}{2},0,\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 1\text{ fixed torus},\tag{3.53}$$

$$\left\{\left(x,y,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},\quad 1\text{ fixed torus}.\tag{3.54}$$

For θ_1 and $\theta_1\theta_2$ one finds also four fixed tori each in a completely analogous way. So, altogether there are 12 fixed tori. Now, a single projection with a non-trivial element of $\mathbb{Z}_2\times\mathbb{Z}_2$ projects out moduli needed for deforming the lattice into the SO(12) lattice. The orbifolds are disconnected.

The computation using the Lefschetz fixed point theorem (3.11) is similar to the last case in the previous example. A basis for the $(1-\theta_2)\,\Lambda$ lattice is

$$(1-\theta_2)\,\Lambda:\quad\begin{matrix}(0,0,2,0,0,0)\,,\\(0,0,0,2,0,0)\,,\\(0,0,0,0,2,0)\,,\\(0,0,0,0,0,2)\,,\end{matrix}\tag{3.55}$$

where the first basis vector originates e.g. from the $\mathrm{SO}(6)^2$ lattice vector $(1,0,1,0,0,0)$. The volume of the fundamental cell is 16. The lattice normal to the invariant lattice has the following basis:

$$N:\quad\begin{matrix}(0,0,1,0,-1,0)\,,\\(0,0,1,0,1,0)\,,\\(0,0,0,1,0,-1)\,,\\(0,0,0,1,0,1)\,.\end{matrix}\tag{3.56}$$

The volume of the fundamental cell of N is four. Hence, the number of fixed tori under θ_1 is $16/4=4$. The computation for θ_1 and $\theta_1\theta_2$ is analogous and yields four fixed tori for each of them.

The points which are fixed under any two non-trivial elements of $\mathbb{Z}_2\times\mathbb{Z}_2$ can be written as

$$(a_1,b_1,a_2,b_2,a_3,b_3)\,,\tag{3.57}$$

where (a_1,a_2,a_3) and (b_1,b_2,b_3) can be any of the following four triplets

$$(0,0,0)\,,\ \ (1,0,0)\,,\ \ \left(\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}\right).\tag{3.58}$$

This gives 16 such points which is four times the number of tori fixed under a single non-trivial $\mathbb{Z}_2\times\mathbb{Z}_2$ element. The Euler number is

$$\chi=24,\tag{3.59}$$

which is twice the number of all fixed tori.

3.4 $SU(3)^3$ lattice

Here, we give a second example for which the total number of fixed tori differs from half the Euler number. The basis for the compactification lattice is given by simple roots of $SU(3)^3$

$$\begin{aligned} e_1 &= \left(\sqrt{2},0,0,0,0,0\right), \\ e_2 &= \left(-\frac{1}{\sqrt{2}},0,0,\sqrt{\frac{3}{2}},0,0\right), \\ e_3 &= \left(0,\sqrt{2},0,0,0,0\right), \\ e_4 &= \left(0,-\frac{1}{\sqrt{2}},0,0,\sqrt{\frac{3}{2}},0\right), \\ e_5 &= \left(0,0,\sqrt{2},0,0,0\right), \\ e_6 &= \left(0,0,-\frac{1}{\sqrt{2}},0,0,\sqrt{\frac{3}{2}}\right). \end{aligned} \tag{3.60}$$

For the above $SU(3)^3$ lattice one finds four fixed tori per $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist. For instance, the θ_2 fixed tori are given by

$$\left\{(x,y,0,0,0,0) \,\middle|\, x,y \in \mathbb{R}^2/\tilde{\Lambda}^2/\sqrt{2}\right\}, \quad 1 \text{ fixed torus}, \tag{3.61}$$

$$\left\{\left(x,y,\frac{1}{\sqrt{2}},0,0,0\right) \,\middle|\, x,y \in \mathbb{R}^2/\tilde{\Lambda}^2/\sqrt{2}\right\}, \quad 1 \text{ fixed torus}, \tag{3.62}$$

$$\left\{\left(x,y,-\frac{1}{2\sqrt{2}},0,0,\frac{1}{2}\sqrt{\frac{3}{2}}\right) \,\middle|\, x,y \in \mathbb{R}^2/\tilde{\Lambda}^2/\sqrt{2}\right\}, \quad 1 \text{ fixed torus}, \tag{3.63}$$

$$\left\{\left(x,y,\frac{1}{2\sqrt{2}},0,0,\frac{1}{2}\sqrt{\frac{3}{2}}\right) \,\middle|\, x,y \in \mathbb{R}^2/\tilde{\Lambda}^2/\sqrt{2}\right\}, \quad 1 \text{ fixed torus}. \tag{3.64}$$

In a similar fashion one finds four fixed tori also for the other two non-trivial elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ yielding to a total number of twelve fixed tori.

In the following we compute the number of tori invariant under the action of θ_2 using the Lefschetz fixed point theorem (3.11). The lattice $(1-\theta_2)\,\Lambda$ is generated by

$$(1-\theta_2)\,\Lambda: \quad \begin{matrix} \left(0,0,2\sqrt{2},0,0,0\right), \\ \left(0,0,0,2\sqrt{\frac{3}{2}},0,0\right), \\ \left(0,0,0,0,2\sqrt{\frac{3}{2}},0\right), \\ \left(0,0,-\sqrt{2},0,0,2\sqrt{\frac{3}{2}}\right). \end{matrix} \tag{3.65}$$

Hence, the induced metric is

$$\begin{pmatrix} 8 & 0 & 0 & -4 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ -4 & 0 & 0 & 8 \end{pmatrix},$$

with determinant $36 \cdot 48$.

The lattice N normal to the invariant lattice is

$$N: \quad \begin{matrix} \left(0,0,\sqrt{2},0,0,0\right), \\ \left(0,0,0,2\sqrt{\tfrac{3}{2}},0,0\right), \\ \left(0,0,0,0,2\sqrt{\tfrac{3}{2}},0\right), \\ \left(0,0,-\tfrac{\sqrt{2}}{2},0,0,\sqrt{\tfrac{3}{2}}\right), \end{matrix} \tag{3.66}$$

with the induced metric

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix},$$

whose determinant is 36·3. The number of fixed tori is the square root of the ratio of the two determinants which is four in the considered case. Taking into account also the other $\mathbb{Z}_2 \times \mathbb{Z}_2$ twists we find 12 fixed tori altogether.

There are eight points on the lattice which are invariant under θ_1 and θ_2. They are of the form that the last three entries of their position vector vanish whereas each of the first three entries can be either zero or $\sqrt{2}/4$. Doing similar countings also for the other combinations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements leads finally to the Euler number,

$$\chi = 12, \tag{3.67}$$

which differs from twice the number of fixed tori.

4. Standard embedding

Now, we consider heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ theory compactified on the previously discussed orbifolds. The massless spectrum in four dimensions does not contain winding or momentum modes (away from the selfdual point). To a large extent, the calculation of the spectrum is independent of the compactification lattice. The lattice enters via its fixed point structure as will be seen shortly.

In this section, we focus on the standard embedding of the orbifold action into the gauge group. In that case the spin connection is related to the gauge connection. This implies that the index of the Dirac operator is fixed by the Euler number which therefore equals twice the number of generations [10, 11].

First, let us summarise what is known from previous calculations [42, 8]. The untwisted sector provides geometric moduli, vector multiplets containing the gauge fields for an unbroken

$$\mathrm{E}_6 \times \mathrm{U}(1)^2 \times \mathrm{E}_8' \tag{4.1}$$

symmetry. Further, there are three chiral multiplets transforming in the **27** and three chiral multiplets in the $\overline{\mathbf{27}}$ of E_6 (we suppress U(1) charges). In addition there are six singlets.

Each non-trivial element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ gives rise to twisted sector states localised to the corresponding fixed tori. Fields appearing in the bulk of a fixed torus are even under the $\mathbb{Z}_2$ element leaving that torus invariant. The remaining $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements relate field values at a point to values at an image point and act as projections if the argument of the field is a fixed point.

The gauge group in the bulk of the fixed tori is

$$\mathrm{E}_7 \times \mathrm{SU}(2) \times \mathrm{E}_8'.$$

The twisted sector gives rise to bulk matter (hypermultiplets) transforming in the (**56**,**1**) plus (**1**,**2**) of $\mathrm{E}_7 \times \mathrm{SU}(2)$ (w.r.t. the hidden E_8' all matter fields are singlets).

The remaining $\mathbb{Z}_2$ projection reduces the bulk gauge group to the group unbroken in four dimensions (4.1). The projections on the twisted matter depend on the compactification lattice[2]. Let us first state what happens in the well studied case that the underlying T^6 factorises into three T^2 factors [42, 8]. In that case the hypermultiplet in the (**56**,**1**) plus (**1**,**2**) is projected to a chiral multiplet in the $\overline{\mathbf{27}}$ of E_6 and five more chiral multiplets which are E_6 singlets. Since this happens for each fixed torus, the number of generations is equal to the number of fixed tori, which is 48 in the considered case.

On the other hand, the net number of generations follows from an index theorem to be half the Euler number. So, the statement that the number of generations equals the number of fixed tori cannot be true in general (not even in $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds) since we have seen that, in certain cases, the two numbers do not agree. (For the example considered above the Euler number is 96 and the result is consistent.)

In the following we will have a closer look at the projections of bulk matter of a θ_i fixed torus, focusing on matter coming from the θ_i twisted sector. First, we study the example from section 3.1 (compactification lattice equals SO(12) root lattice) for which the Euler number is twice the number of fixed tori.

The θ_2 twisted sector provides eight hypermultiplets in the (**56**,**1**) + (**1**,**2**) of $\mathrm{E}_7 \times \mathrm{SU}(2)$, one on each of the 8 fixed tori

$$\left\{(x, y, 0, 0, 0, 0) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}, \quad \text{1 fixed torus,} \tag{4.2}$$

$$\left\{\left(x, y, \underline{\frac{1}{2}, \frac{1}{2}}, 0, 0\right) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}, \quad \text{6 fixed tori,} \tag{4.3}$$

$$\left\{\left(x, y, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \,\middle|\, x, y \in \mathbb{R}^2\Lambda^2\right\}, \quad \text{1 fixed torus.} \tag{4.4}$$

Now, we discuss at which points θ_1 invariance provides projection conditions on these hypermultiplets. First, consider the fixed tori of the form

$$\left\{(x, y, 0, 0, 0, 0) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\},$$
$$\left\{\left(x, y, \frac{1}{2}, \frac{1}{2}, 0, 0\right) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\},$$

[2]In technical terms: The centraliser [10] of a space group element depends on the compactification lattice.

$$\left\{\left(x,y,0,0,\frac{1}{2},\frac{1}{2}\right)\,|\,x,y\in\mathbb{R}^2/\Lambda^2\right\},$$
$$\left\{\left(x,y,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\,|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}. \tag{4.5}$$

Taking into account SO(12) lattice shifts θ_1 maps a point on one of these tori to a point on the same torus with $(x,y)\to-(x,y)$. Projection conditions thus arise at the points for which

$$(x,y)=-(x,y)\ \mathrm{mod}\ \Lambda^2,$$

which is satisfied for the following four points

$$(0,0),\ \left(\frac{1}{2},-\frac{1}{2}\right),\ \left(\frac{1}{2},\frac{1}{2}\right),\ (1,0). \tag{4.6}$$

Imposing invariance under θ_1 at these points reduces the hypermultiplets to chiral ones in $\overline{\mathbf{27}}$ of E_6 and singlets. So, each of the four tori in (4.5) gives rise to one generation.

Points on the remaining four fixed tori transform as

$$\left(x,y,\underline{\frac{1}{2}},0,\underline{\frac{1}{2}},0\right)\xrightarrow{\theta_1}\left(-x,-y,\underline{-\frac{1}{2}},0,\underline{\frac{1}{2}},0\right)\equiv\left(-x+1,-y,\underline{\frac{1}{2}},0,\underline{\frac{1}{2}},0\right), \tag{4.7}$$

where in the last step equivalence under shifts by the SO(12) lattice vectors

$$\left(1,0,0,\underline{1,0},0,0\right) \tag{4.8}$$

has been employed. Thus θ_1 maps a point on a fixed torus to itself if (x,y) is at one of the following four points[3]

$$\left(\frac{1}{2},0\right),\ \left(\frac{3}{2},\frac{1}{2}\right),\ \left(\frac{3}{2},-\frac{1}{2}\right),\ \left(\frac{3}{2},0\right). \tag{4.9}$$

So, also each hypermultiplet on these remaining fixed tori gives rise to a chiral multiplet transforming in the $\overline{\mathbf{27}}$ of E_6 and singlets.

Repeating our discussion for the remaining two twist sectors, we find that the number of generations is equal to the number of fixed tori, i.e. 24. Since in the considered case the Euler number is 48, our result is consistent with the index theorem.

As a next case we study the example of section 3.2. In this case the Euler number differs from twice the number of fixed tori. First, let us look at the θ_2 twisted sector. Again, there is a hypermultiplet in $(\mathbf{56},\mathbf{1})+(\mathbf{1},\mathbf{2})$ on each of the θ_2 fixed tori (3.27)–(3.30). For points on the first two fixed tori (3.27) and (3.28) the situation is the same as in the previous example in that these points are mapped onto points in the same torus with four

[3]Considering an effective six dimensional orbifold GUT it looks a bit strange that we have two sets with four fixed points in each set. However, such an effective six dimensional orbifold GUT is obtained by taking the volume of the fixed torus to be much larger than the volume of the remaining compact space (by tuning metric moduli). Then shifts of the form (4.8) appear as an effective symmetry $x\to x+1$ resulting in a compactification torus whose lattice is generated by $(1,0)$ and $(0,1)$. On that smaller torus some fixed points become indistinguishable, and the effective six dimensional orbifold GUT is compactified on $T^2/\mathbb{Z}_2$.

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points being invariant. Hence, these two fixed tori result in two chiral multiplets in the $\overline{\mathbf{27}}$ of E_6 plus singlets. The same happens for fixed tori of the form

$$\left\{\left(x, y, 0, 0, \frac{1}{2}, \frac{1}{2}\right) \mid x, y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{4.10}$$

$$\left\{\left(x, y, 0, 0, \frac{1}{2}, -\frac{1}{2}\right) \mid x, y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{4.11}$$

giving rise to two more chiral multiplets in the $\overline{\mathbf{27}}$ and singlets. For the remaining four fixed tori

$$\left\{\left(x, y, 0, \frac{1}{2}, 0, \underline{\frac{1}{2}}\right) \mid x, y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{4.12}$$

$$\left\{\left(x, y, 0, \frac{1}{2}, 0, \underline{-\frac{1}{2}}\right) \mid x, y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{4.13}$$

the situation is different. Points in tori of the form (4.12) are mapped to points on tori of the form (4.13). This means that projection conditions identify hypermultiplets, leading to two massless hypermultiplets in four dimensions. In $N = 1$ language, one hypermultiplet decomposes into a chiral multiplet in the $\mathbf{27}$, a chiral multiplet in the $\overline{\mathbf{27}}$ and singlets. Hence, we get two generations and two anti-generations from the twisted states on (4.12) and (4.13). By employing the symmetry between the two SO(6) factors in the compactification lattice, we deduce that the θ_1 twisted sector also gives six chiral multiplets in the $\mathbf{27}$ and two chiral multiplets in the $\overline{\mathbf{27}}$, with a net number of four generations.

$\theta_1\theta_2$ fixed tori are listed in (3.36)–(3.36). Points on each of these tori are mapped onto points on the same torus. Hence, the number of generations coming from the $\theta_1\theta_2$ twisted sector is four.

In summary, for the example of section 3.2 we obtain 18 generations and six anti-generations, from twisted sectors. This relates to the topological data as follows: The net number of generations equals half the Euler number and the surplus in the number of fixed tori provides pairs of generations and anti-generations.

In the example of section 3.3, points on a fixed torus are always mapped on points of the same torus. The number of generations is equal to the number of fixed tori, which is the same as half the Euler number.

Finally, we discuss the example of section 3.4. The tori fixed under the action of θ_2 are listed in (3.61)–(3.64). Points on the first two fixed tori, (3.61) and (3.62), are mapped on points of the same torus by θ_1. Hence, these two tori give rise to two generations in four dimensions. On the other hand, θ_1 maps points on the fixed torus (3.63) to points on (3.64). These two tori result in a generation and an anti-generation. Repeating the discussion for the other orbifold twist, we find from twisted sectors nine generations and three anti-generations. The net number six coincides with half the Euler number.

We summarise our examples in table 1, where we also added the untwisted sector providing three generations and three anti-generations. In addition, the number of fixed tori and the Euler number is listed in table 2.

These results can be confirmed by deriving projections from modular invariance as was done in [12] and extended in [39] for the case of fixed tori and non factorisable lattices. The

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Lattice from section:		(generations , anti-generations)	net number of generations
3.1	(SO(12))	(27,3)	24
3.2	$(SO(6)^2$-A)	(19,7)	12
3.3	$(SO(6)^2$-B)	(15,3)	12
3.4	$(SU(3)^3)$	(12,6)	6

Table 1: Number of generations for standard embedding.

Lattice from section:		fixed tori	Euler number
3.1	(SO(12))	24	48
3.2	$(SO(6)^2$-A)	20	24
3.3	$(SO(6)^2$-B)	12	24
3.4	$(SU(3)^3)$	12	12

Table 2: Number of fixed tori and Euler number.

strategy is to split the partition function into various pieces corresponding to different twist sectors and different insertions of orbifold group elements into the trace. Such splitting mixes under modular transformations and hence modular invariance fixes numerical factors in front of some terms. The number of generations and anti-generations can be deduced by identifying contributions from chiral and anti-chiral fermions to the trace.

A subtlety for non factorisable compactifications is the existence of contributions in which winding and momentum sums are not over mutually dual lattices. For instance, in the trace over untwisted states with a θ_i insertion only windings on an invariant sublattice contribute, whereas momenta are constrained to an invariant sublattice of the dual compactification lattice. The latter is the dual of a θ_i projected lattice which differs from the θ_i invariant lattice in the non factorisable case. Modular invariance relates that contribution to the θ_i twisted sector. There, windings are constrained to the θ_i projected lattice, since strings have to close only up to a θ_i identification. (The invariant lattice is a sublattice of the projected lattice.) Momenta, on the other hand, take values on the dual of the θ_i invariant lattice which spans the θ_i fixed torus.

Thus we observe that the two contributions discussed above are related by interchanging winding and momenta and dualising the involved lattices. This can be achieved by Poisson resummation and hence corresponds to modular transformations. Numerical factors equal to ratios of the volumes of invariant and projected lattices appear, and this explains the lattice dependence in the number of families and anti-families. (We have carried out a detailed analysis of all cases discussed in this paper and found the correct numerical factors.)

Another subtlety which we did not mention so far is the ambiguity in the choice of sign for some of the different contributions to the partition function. This corresponds to the inclusion of discrete torsion [43, 42, 44]. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case the terms in question are traces over θ_i twisted sectors with θ_j $(i \neq j)$ insertions. In these traces chiral and anti-chiral fermions contribute with opposite signs. Hence, switching on and off discrete

torsion just swaps the number of families with the number of anti-families (see [42]). As discussed in [44], for the factorisable case, this corresponds to replacing the Hodge diamond by its mirror. Our discussion suggests that the cases with and without discrete torsion are related by mirror symmetry also if the underlying T^6 does not factorise.

Finally, a comment about non-standard embeddings is in order. Since we discussed each twisted sector separately it should be straightforward to modify our considerations for the other consistent choices of the orbifold embedding into the gauge group (listed in [17]).

5. A three generation model

In this section we present a concrete three generation model with an unbroken SO(10) gauge group. The orbifold will be standard embedded and from the previous section it is clear that we need to add Wilson lines [9]. The number of models with Wilson lines is large and, in order to ease our task, we seek some intuition.

First, we will focus on models with SO(10) gauge group. This simplifies the counting of generations since one generation corresponds to a 16 dimensional representation. From a phenomenological perspective, the construction of SO(10) models can be viewed as a step towards obtaining realistic models. This is suggested by the free fermionic models discussed in section two, as well as the geometric picture [8, 16].

Secondly, we borrow some more intuition from the free fermionic construction by looking for models where each twisted sector provides one generation. For $\mathbb{Z}_3$ orbifolds, models where the number of generations is associated to the number of complex extra dimensions [9] as well as to multiplicities of twisted sectors [45] have been constructed. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case, however, most of the known three generation models [8] do not exhibit such a structure. The untwisted spectrum is always non chiral and trying to associate the number of generations to the complex number of extra dimensions is hopeless.

So, it remains to look for models where the three twisted sectors contribute one generation each. As we will argue now, the fact whether or not the underlying T^6 factorises into the product of three T^2 is essential in that context. In order to break the E_6 gauge symmetry down to SO(10) one needs to turn on a Wilson line removing the 16 dimensional spinors of SO(10) from the adjoint of E_6. In the case of a factorisable T^6, the cycle along which this Wilson line is turned on will be invariant under one of the three non trivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements. Within the corresponding twisted sector, the Wilson line will impose a projection removing the 16 dimensional representations of SO(10). Thus that sector cannot contribute to the number of generations.[4]

For non factorisable tori, on the other hand, there exist cycles which are invariant under none of the three non trivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements. Turning on the Wilson line breaking E_6 to SO(10) on such a cycle does not necessarily remove **16** representations from one of the twisted sectors.

[4] Our argument can be evaded by a more baroque embedding of SO(10) into E_6. An example is reported in [46]. We thank Patrick Vaudrevange for drawing our attention to that possibility.

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In the explicit calculation one has to determine the contribution from each fixed torus to the spectrum. So, as a final simplification, we focus on models with a minimal number of fixed tori.

Therefore, we consider the $SO(6)^2$ lattice of section 3.3 where we will indeed find a three generation model with SO(10) gauge symmetry. Before giving the details of that model, let us discuss consistency conditions for the Wilson lines. For the present considerations it is useful to view the Wilson line as a vacuum expectation value for an internal gauge field component, A_i, where the index labels directions along lattice vectors (3.50) (see also (2.7) for the notation). For discrete Wilson lines the value of A_i can differ from its orbifold image only by $E_8 \times E_8$ root lattice vectors [9]. In order to find the resulting consistency condition we state the orbifold action on the vectors generating the $SO(6)^2$ lattice (3.50)

$$\theta_1 : \begin{array}{l} e_1 \to -e_1, \\ e_2 \to -e_3, \\ e_3 \to -e_2, \\ e_4 \to -e_4, \\ e_5 \to -e_6, \\ e_6 \to -e_5, \end{array} \qquad \theta_2 : \begin{array}{l} e_1 \to e_1 + e_2 + e_3, \\ e_2 \to -e_2, \\ e_3 \to -e_3, \\ e_4 \to e_4 + e_5 + e_6, \\ e_5 \to -e_5, \\ e_6 \to -e_6. \end{array} \tag{5.1}$$

Hence we find the following consistency conditions

$$2A_i, \;\; A_2 + A_3, \;\; A_5 + A_6 \in \Lambda_{E_8 \times E_8}, \;\; i = 1, \ldots, 6, \tag{5.2}$$

where $\Lambda_{E_8 \times E_8}$ denotes the $E_8 \times E_8$ root lattice. The other condition which has to be satisfied comes from modular invariance (see e.g. [12]).

Adopting the notation of e.g. [8] we characterise the standard embedding by the shift vectors

$$V_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0^6\right)\left(0^8\right) \quad , \quad V_2 = \left(0, \frac{1}{2}, -\frac{1}{2}, 0^5\right)\left(0^8\right). \tag{5.3}$$

Further, we turn on the following set of consistent Wilson lines

$$A_1 = \left(0^8\right)\left(0^3, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\right), \tag{5.4}$$

$$A_2 = A_3 = \left(0^7, 1\right)\left(1, 0^7\right), \tag{5.5}$$

$$A_4 = \left(0^8\right)\left(0, -\frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0\right), \tag{5.6}$$

$$A_5 = A_6 = \left(0^8\right)\left(0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0^3\right). \tag{5.7}$$

For the computation of the twisted sectors one has to identify the space group element leaving a given torus invariant. For instance, the torus (3.51) is fixed under the rotation by θ_2, whereas for the tori (3.52), (3.53), (3.54) the rotation by θ_2 has to be supplemented by a shift with e_3, e_6, $e_3 + e_6$, respectively. With these remarks the calculation of the massless spectrum follows straightforward modifications of standard techniques and we refrain from giving a detailed step by step presentation. Instead, we just report the result.

The metric and B field moduli are g_{IJ} and b_{IJ} with $(I,J) = (1,2)$, $(3,4)$ or $(5,6)$, where the indices I, J label Cartesian coordinates. We choose these moduli to be different from their selfdual values in order not to have to report more massless states than necessary.

The untwisted spectrum gives rise to an $N = 1$ vector multiplet in the adjoint of

$$\mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SU}(2)^8, \tag{5.8}$$

where the SO(10) and U(1) factors come from the first E_8 factor whereas the SU(2) factors originate from the second (hidden) E_8 factor. The three U(1) factors are generated by the first three Cartan operators of the first E_8: H_1, H_2, H_3. The remaining Cartan generators of the first E_8, $H_3, \ldots, H_8$, form the Cartan Subalgebra of SO(10). Denoting the Cartan generators of the second E_8 by $H_9, \ldots, H_{16}$ we arrange the eight SU(2) factors according to the following order of their Cartan generators:

$$H_9+H_{16}, H_9-H_{16}, H_{10}+H_{11}, H_{10}-H_{11}, H_{12}+H_{13}, H_{12}-H_{13}, H_{14}+H_{15}, H_{14}-H_{15}. \tag{5.9}$$

Below we will characterise representations of the unbroken gauge group (5.8) by an ordered nonet containing the dimensionality of the SO(10) representation followed by the dimensionalities of the $\mathrm{SU}(2)^8$ representation. A sequence of n consecutive entries being $\mathbf{1}$ will be abbreviated by $\mathbf{1}^n$. Further, a triple subscript will denote the U(1) charges.[5]

In the **untwisted sector** one finds chiral multiplets transforming as

$$\left(\mathbf{10};\mathbf{1}^8\right)_{1,0,0} + \left(\mathbf{1};\mathbf{1}^8\right)_{0,1,1} + \left(\mathbf{1};\mathbf{1}^8\right)_{0,1,-1} + \text{Charge Conjugate}, \tag{5.10}$$

$$\left(\mathbf{10};\mathbf{1}^8\right)_{0,1,0} + \left(\mathbf{1};\mathbf{1}^8\right)_{1,0,1} + \left(\mathbf{1};\mathbf{1}^8\right)_{1,0,-1} + \text{Charge Conjugate}, \tag{5.11}$$

$$\left(\mathbf{10};\mathbf{1}^8\right)_{0,0,1} + \left(\mathbf{1};\mathbf{1}^8\right)_{1,1,0} + \left(\mathbf{1};\mathbf{1}^8\right)_{1,-1,0} + \text{Charge Conjugate}, \tag{5.12}$$

where (5.10), (5.11), (5.12) correspond to the first, second and third complex plane, respectively.

Now, we turn to the $\boldsymbol{\theta_1}$ **twisted sector**. The four θ_1 fixed tori are

$$\left\{(0,0,0,0,x,y) \,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{5.13}$$

$$\left\{\left(\frac{1}{2},0,\frac{1}{2},0,x,y\right) \,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{5.14}$$

$$\left\{\left(0,\frac{1}{2},0,\frac{1}{2},x,y\right) \,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{5.15}$$

$$\left\{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},x,y\right) \,\middle|\, x,y \in \mathbb{R}^2/\Lambda^2\right\}. \tag{5.16}$$

On each of these tori massless chiral multiplets are localised. Below, we list the representations in which they transform:

[5]For the impatient reader: The important part of the result is that we find one 16 dimensional representation of SO(10) in each twisted sector. These representations are localised on fixed tori situated at the origin of the remaining four directions.

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- torus (5.13):

$$\begin{aligned}&\left(\mathbf{16};\mathbf{1}^8\right)_{0,0,\frac{1}{2}}+\left(\mathbf{10};\mathbf{1}^8\right)_{-\frac{1}{2},-\frac{1}{2},0}+\left(\mathbf{1};\mathbf{1}^8\right)_{\frac{1}{2},\frac{1}{2},1}+\left(\mathbf{1};\mathbf{1}^8\right)_{\frac{1}{2},\frac{1}{2},-1}+\\&2\left(\mathbf{1};\mathbf{1}^8\right)_{\frac{1}{2},-\frac{1}{2},0}+2\left(\mathbf{1};\mathbf{1}^8\right)_{-\frac{1}{2},\frac{1}{2},0},\end{aligned}\tag{5.17}$$

- torus (5.14):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^5\right)_{-\frac{1}{2},\frac{1}{2},0}+\left(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^4\right)_{-\frac{1}{2},\frac{1}{2},0}+\\&\left(\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}\right)_{\frac{1}{2},-\frac{1}{2},0}+\left(\mathbf{1};\mathbf{1}^5,\mathbf{2},\mathbf{1},\mathbf{2}\right)_{\frac{1}{2},-\frac{1}{2},0},\end{aligned}\tag{5.18}$$

- torus (5.15):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{1}^3,\mathbf{2},\mathbf{1}\right)_{\frac{1}{2},-\frac{1}{2},0}+\left(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1}^3,\mathbf{2},\mathbf{1}\right)_{\frac{1}{2},-\frac{1}{2},0}+\\&\left(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^3,\mathbf{2}\right)_{-\frac{1}{2},\frac{1}{2},0}+\left(\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{1}^3,\mathbf{2}\right)_{-\frac{1}{2},\frac{1}{2},0},\end{aligned}\tag{5.19}$$

- torus (5.16):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{1}^5,\mathbf{2},\mathbf{1}\right)_{\frac{1}{2},-\frac{1}{2},0}+\left(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^5,\mathbf{2}\right)_{\frac{1}{2},-\frac{1}{2},0}+\\&\left(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^3\right)_{-\frac{1}{2},\frac{1}{2},0}+\left(\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^2\right)_{-\frac{1}{2},\frac{1}{2},0}.\end{aligned}\tag{5.20}$$

So, there is one generation on the torus (5.13).

In the $\boldsymbol{\theta_2}$ **twisted sector** one finds chiral multiplets localised on the tori (3.51)–(3.54). The representations are

- torus (3.51):

$$\begin{aligned}&\left(\mathbf{16};\mathbf{1}^8\right)_{\frac{1}{2},0,0}+\left(\mathbf{10};\mathbf{1}^8\right)_{0,-\frac{1}{2},-\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^8\right)_{1,\frac{1}{2},\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^8\right)_{-1,\frac{1}{2},\frac{1}{2}}+\\&2\left(\mathbf{1};\mathbf{1}^8\right)_{0,\frac{1}{2},-\frac{1}{2}}+2\left(\mathbf{1};\mathbf{1}^8\right)_{0,-\frac{1}{2},\frac{1}{2}},\end{aligned}\tag{5.21}$$

- torus (3.52):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{2},\mathbf{1}^6\right)_{0,-\frac{1}{2},-\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{2},\mathbf{1}^4\right)_{0,-\frac{1}{2},-\frac{1}{2}}+\\&\left(\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{2},\mathbf{1}^2\right)_{0,\frac{1}{2},\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^6,\mathbf{2},\mathbf{2}\right)_{0,\frac{1}{2},\frac{1}{2}},\end{aligned}\tag{5.22}$$

- torus (3.53):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^5,\mathbf{2}\right)_{0,-\frac{1}{2},\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^2\right)_{0,-\frac{1}{2},\frac{1}{2}}+\\&\left(\mathbf{1};\mathbf{2},\mathbf{1}^5,\mathbf{2},\mathbf{1}\right)_{0,\frac{1}{2},-\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^2\right)_{0,\frac{1}{2},-\frac{1}{2}},\end{aligned}\tag{5.23}$$

- torus (3.54):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{1}^6,\mathbf{2}\right)_{0,\frac{1}{2},\frac{1}{2}}+\left(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^4,\mathbf{2},\mathbf{1}\right)_{0,\frac{1}{2},\frac{1}{2}}+\\&\left(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1}^2,\mathbf{2},\mathbf{1}^2\right)_{0,-\frac{1}{2},\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{2},\mathbf{1}^3\right)_{0,-\frac{1}{2},\frac{1}{2}}.\end{aligned}\tag{5.24}$$

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There is one generation coming from the θ_2 twisted sector localised on the torus (3.51).

$\theta_1\theta_2$ twisted sector states are localised on the following four tori:

$$\left\{(0,0,x,y,0,0)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}, \tag{5.25}$$

$$\left\{\left(\frac{1}{2},0,x,y,\frac{1}{2},0\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}, \tag{5.26}$$

$$\left\{\left(0,\frac{1}{2},x,y,0,\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}, \tag{5.27}$$

$$\left\{\left(\frac{1}{2},\frac{1}{2},x,y,\frac{1}{2},\frac{1}{2}\right)\,\middle|\,x,y\in\mathbb{R}^2/\Lambda^2\right\}. \tag{5.28}$$

The corresponding massless chiral multiplets transform as:

- torus (5.25):

$$\begin{aligned}&\left(\mathbf{16};\mathbf{1}^8\right)_{0,\frac{1}{2},0}+\left(\mathbf{10};\mathbf{1}^8\right)_{-\frac{1}{2},0,-\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^8\right)_{\frac{1}{2},1,\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^8\right)_{\frac{1}{2},-1,\frac{1}{2}}+\\&2\left(\mathbf{1};\mathbf{1}^8\right)_{\frac{1}{2},0,-\frac{1}{2}}+2\left(\mathbf{1};\mathbf{1}^8\right)_{-\frac{1}{2},0,\frac{1}{2}},\end{aligned} \tag{5.29}$$

- torus (5.26):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{1}^2,\mathbf{2},\mathbf{1}^4\right)_{\frac{1}{2},0,\frac{1}{2}}+\left(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}^5\right)_{\frac{1}{2},0,\frac{1}{2}}+\\&\left(\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{1}^2,\mathbf{2}\right)_{-\frac{1}{2},0,-\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^5,\mathbf{2},\mathbf{2},\mathbf{1}\right)_{-\frac{1}{2},0,-\frac{1}{2}},\end{aligned} \tag{5.30}$$

- torus (5.27):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^5\right)_{-\frac{1}{2},0,\frac{1}{2}}+\left(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^4\right)_{\frac{1}{2},0,-\frac{1}{2}}+\\&\left(\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}\right)_{-\frac{1}{2},0,\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^5,\mathbf{2},\mathbf{1},\mathbf{2},\right)_{\frac{1}{2},0,-\frac{1}{2}},\end{aligned} \tag{5.31}$$

- torus (5.28):

$$\begin{aligned}&\left(\mathbf{1};\mathbf{2},\mathbf{2},\mathbf{1}^6\right)_{-\frac{1}{2},0,-\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{2},\mathbf{1}^4\right)_{\frac{1}{2},0,\frac{1}{2}}+\\&\left(\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{2},\mathbf{1}^2\right)_{\frac{1}{2},0,\frac{1}{2}}+\left(\mathbf{1};\mathbf{1}^6,\mathbf{2},\mathbf{2}\right)_{-\frac{1}{2},0,-\frac{1}{2}}.\end{aligned} \tag{5.32}$$

There is one generation localised on the fixed torus (5.25).

The important part of the above spectrum is that there is one generation from each twisted sector, yielding the observed value of three generations in total. We have listed also all the other details of the spectrum, because these enable one to perform non trivial consistency checks. In the case at hand, all factors in the unbroken gauge group are anomaly free. For the three U(1) factors this is a non trivial statement and we have checked that the corresponding 36 triangle diagrams indeed vanish.

To conclude, let us summarise how the construction worked. By the standard embedded orbifold one of the E_8 factors is broken to E_6. The Wilson line (5.5) is responsible for breaking E_6 down to SO(10), whereas all the remaining Wilson lines remove **16** or $\overline{\mathbf{16}}$

representations from all but one of the four fixed tori in each twisted sector. Since we did not want to break SO(10) further, we have chosen them such that they break only the hidden sector gauge group. It is conceivable that, alternatively, one can choose some of the additional Wilson lines such that SO(10) is broken further to its Standard Model subgroup and, simultaneously, keep the three 16 dimensional representations providing the Standard Model matter plus three right handed neutrinos. This construction is to be carried out in the near future [47].

6. Conclusions and outlook

The present paper started by summarising the free fermionic constructions of four dimensional heterotic string theories. We argued that the impressive success in reproducing essential features of real particle physics should encourage us to seek similar features in $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactifications of heterotic string theory. In particular, free fermionic constructions suggest that the underlying six-dimensional compactification lattice should be the SO(12) root lattice.

Motivated by the above observations, we studied the geometric properties of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of several non factorisable six-tori. In great detail, we discussed how to compute the number of fixed tori and the Euler number. Subsequently, we related these data to the particle spectrum, if the orbifold is standard embedded into the gauge group. Without Wilson lines there are no three generation models.

By introducing discrete Wilson lines [9], one can break the gauge group and, simultaneously, reduce the number of generations. We observed that for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on non factorisable tori one can easily find models where each of the three twisted sectors contributes exactly one generation, whereas this is difficult in the case that the underlying T^6 factorises into a product of three T^2. This supports the conjecture that free fermionic constructions are related to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of non factorisable tori since also, in free fermionic models, cases with the same distribution of the three families exist. We illustrated the discussion by the explicit presentation of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of T^6 whose compactification lattice is a root lattice of $\mathrm{SO}(6)^2$.

There are several directions to be explored in future work. To make explicitly contact to the quasi-realistic free fermionic models [24 – 28], one should consider the SO(12) root lattice as a compactification lattice. Realising these models in the orbifold language would not only improve our understanding of the correspondence between the two schemes, but also provide insight to deformations away from the free fermionic point. Orbifold constructions would benefit as well, since the free fermionic approach is suitable for a classification of models [35, 36]. One can then envision a one-to-one map between the string vacuum in the fermionic and orbifold representations. Such a map will be instrumental in elucidating the phenomenological and cosmological properties of specific string vacua in this class.

Moreover, we have seen that, in general, non factorisable compactifications are promising in their own right. For instance, by modifying the set of Wilson lines in our construction, it should be not too difficult to find a three generation standard like model for which matter is naturally embedded into 16 dimensional representations of an underlying SO(10). That

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guaranties a correct prediction for the hypercharges. Such a model should then undergo further tests checking its relevance for the description of the real world.

Our investigations clearly show that $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of non factorisable six-tori should be well received into the class of phenomenologically promising string vacua.

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